Math 334 Test 2 KEY
Spring 2010
Section: 001

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Date: May 24.

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Honor Code: I will not look at other students' papers as I am taking this exam.

Signature:

1. By using one of the estimates from Picard's proof of the Fundamental Theorem of First Order ODE's, show that there exists a solution $y=\phi(t)$ to the IVP

$$\frac{dy}{dt}=\frac{3t^2}{2}\left(1+y^2\right),\ y(0)=0\,, \tag{0.1}$$

at least throughout the interval

$$t\in\left[-h,h\right]=\left[-1,1\right].^{1} \tag{0.2}$$

Hint: If you do not remember the estimate, do the following to jog your memory. Instead of (0.1), and following Picard, write (for some $h>0$ to be determined) that

$$\phi(t)=\frac{3}{2}\int_0^t s^2\left(1+\phi^2(s)\right)ds,\ \ |t|\le h\,, \tag{0.3}$$

and, so, deduce that

$$|t|\le h\Rightarrow|\phi(t)|\le\frac{3}{2}\int_0^{|t|}\left|s^2\left(1+\phi^2(s)\right)\right|ds=\frac{3}{2}\int_0^{|t|}s^2\left(1+\phi^2(s)\right)ds\le\frac{3}{2}\int_0^{h}s^2\left(1+\phi^2(s)\right)ds\,. \tag{0.4}$$

Now demand that h is the biggest number that is still small enough so that, for some $Y>0$, imposing $|\phi(s)|\le Y$ (for each $s\in\left[-h,h\right]$) on the right hand side of

[1] Note that the solution of (0.1) actually persists throughout the interval $t\in\left(-(\pi)^{1/3},(\pi)^{1/3}\right)\doteq\left(-1.46,1.46\right)$, since the solution has the formula $\tan\left(t^3/2\right)$, and since the domain of (the relevant instance of) the tangent function is $t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

(0.4) certainly ensures that $|\phi(t)| \le Y$ (for each $t \in [-h,h]$) on the left of (0.4). By doing this you will get an h that depends on Y, i.e. you will get an $h(Y)$. Now find

$$h := \max_{Y>0} h(Y). \qquad (0.5)$$

This should be the number indicated in(0.2), i.e. the result of (0.5) should be 1.

10 points

Solution

Demanding $|\phi(s)| \le Y$ (for each $s \in [-h,h]$) on the right hand side of (0.4) gives there that

$$|t| \le h \Rightarrow |\phi(t)| \le \frac{3}{2}\int_0^h s^2\left(1+\phi^2(s)\right)ds \le \frac{3}{2}\int_0^h s^2\left(1+Y^2\right)ds = \frac{h^3}{2}\left(1+Y^2\right). \qquad (0.6)$$

So now we certainly get $|\phi(t)| \le Y$ (for each $t \in [-h,h]$) provided we choose h small enough so that $\frac{h^3}{2}\left(1+Y^2\right) \le Y$, the largest such h accomplishing this being

$$h(Y) = \left(\frac{2Y}{1+Y^2}\right)^{1/3}. \qquad (0.7)$$

So then we get the h indicated in (0.2) by noting that

$$h := \max_{Y>0} h(Y) = \max_{Y>0}\left(\frac{2Y}{1+Y^2}\right)^{1/3} = \left(\frac{2\cdot 1}{1+1^2}\right)^{1/3} = \left(\frac{2}{2}\right)^{1/3} = 1. \qquad (0.8)$$

We could get the maximum indicated in (0.8) by using the relevant tools from calculus.

2. Suppose we have 2 continuous solutions $\phi(t)$ and $\psi(t)$, $t \in [-h,h] = [-1,1]$, to the integral equation indicated in (0.3), i.e. suppose that both

$$\begin{aligned} \phi(t) &= \frac{3}{2}\int_0^t s^2\left(1+\phi^2(s)\right)ds, \text{ and} \\ \psi(t) &= \frac{3}{2}\int_0^t s^2\left(1+\psi^2(s)\right)ds \end{aligned} \qquad (0.9)$$

for each $t \in [-1,1]$. Then we could note that the difference $\phi(t)-\psi(t)$ of these two solutions obeys

$$\begin{aligned} \phi(t)-\psi(t) &= \frac{3}{2}\int_0^t \left[s^2\left(1+\phi^2(s)\right) - s^2\left(1+\psi^2(s)\right)\right] ds = \frac{3}{2}\int_0^t s^2\left(\phi^2(s)-\psi^2(s)\right) ds \\ &= \frac{3}{2}\int_0^t s^2\left(\phi(s)+\psi(s)\right)\left(\phi(s)-\psi(s)\right) ds, \end{aligned} \tag{0.10}$$

and, for $t \in [-1,1]$, we could get the estimate

$$\begin{aligned} \left|\phi(t)-\psi(t)\right| &= \left|\frac{3}{2}\int_0^t s^2\left(\phi(s)+\psi(s)\right)\left(\phi(s)-\psi(s)\right) ds\right| \\ &= \frac{3}{2}\int_0^{|t|} s^2\left|\phi(s)+\psi(s)\right|\left|\phi(s)-\psi(s)\right| ds \\ &\le \frac{3}{2}\int_0^{|t|} \left\{\max_{\tau\in[0,1]} \tau^2\left|\phi(\tau)+\psi(\tau)\right|\right\}\left|\phi(s)-\psi(s)\right| ds \\ &=: \frac{3}{2}\int_0^{|t|} K\left|\phi(s)-\psi(s)\right| ds = \frac{3}{2}K\int_0^{|t|}\left|\phi(s)-\psi(s)\right| ds \end{aligned} \tag{0.11}$$

where evidently

$$0 \le K := \max_{\tau\in[0,1]} \tau^2\left|\phi(\tau)+\psi(\tau)\right| < \infty, \tag{0.12}$$

the last inequality in (0.12) holding because we have a continuous function on the bounded interval $\tau \in [0,1]$. So now define the new function

$$U(t) := \int_0^t \left|\phi(s)-\psi(s)\right| ds \tag{0.13}$$

for each $t \in [-1,1]$, and note that

$$\begin{aligned} &U(t) \ge 0, t \in [0,1], \\ &U(0) = 0, \end{aligned} \tag{0.14}$$

and also then note that (0.11) can then be written as

$$U'(t) = \left|\phi(t)-\psi(t)\right| \le \frac{3}{2}K\int_0^{|t|}\left|\phi(s)-\psi(s)\right| ds = \frac{3}{2}KU(t) \tag{0.15}$$

for $t \in [0,1]$, i.e. we get

$$U'(t) \leq \frac{3}{2}KU(t), t \in [0,1]. \tag{0.16}$$

Use (0.14) and (0.16) to show that

$$U(t) = 0, \tag{0.17}$$

for $t \in [0,1]$ and, so, deduce that $\phi(t) = \psi(t), t \in [0,1]$, i.e. deduce that there is at most one continuous solution to the integral equation (0.3) for $t \in [0,1]$. (You could also show $U(t) = 0$ for $t \in [-1,0]$ by a related but different argument.)

15 points

Solution

From (0.16) we have that, for each $t \in [0,1]$,

$$U'(t) - \frac{3}{2}KU(t) \leq 0, \tag{0.18}$$

and then that

$$e^{-3/2Kt}U'(t) - \frac{3}{2}e^{-3/2Kt}KU(t) \leq 0 \tag{0.19}$$

for $t \in [0,1]$. But then since

$$e^{-3/2Kt}U'(t) - \frac{3}{2}e^{-3/2Kt}KU(t) = \frac{d}{dt}\left(e^{-3/2Kt}U(t)\right), \tag{0.20}$$

(0.19) is

$$\frac{d}{dt}\left(e^{-3/2Kt}U(t)\right) \leq 0, \tag{0.21}$$

again for $t \in [0,1]$. Together with $U(0) = 0$ (see (0.14)), (0.21) gives, for each $t \in [0,1]$,

$$\begin{aligned} e^{-3/2Kt}U(t) = e^{-3/2Kt}U(t) - 0 = e^{-3/2Kt}U(t) - e^{-3/2K\cdot 0}U(0) = e^{-3/2Ks}U(s)\Big|_{s=0}^{s=t} = \\ \int_0^t \frac{d\left(e^{-3/2Ks}U(s)\right)}{ds}ds \leq \int_0^t 0ds \\ = 0, \end{aligned} \tag{0.22}$$

i.e. (0.21) gives

$$e^{-3/2Kt}U(t) \leq 0, \tag{0.23}$$

or, equivalently,

$$U(t) \le 0 \tag{0.24}$$

for each $t \in [0,1]$. From both (0.24) and $U(t) \ge 0$ (from (0.14)), we must have $U(t) = 0$. But then, using (0.13), we have

$$0 = \frac{d}{dt}0 = \frac{d}{dt}U(t) = \frac{d}{dt}\int_0^t |\phi(s) - \psi(s)|\, ds = |\phi(t) - \psi(t)| \tag{0.25}$$

for $t \in [0,1]$, which then gives $\phi(t) = \psi(t)$ for $t \in [0,1]$.

3.

a) Find the value of y_0 so that the following I.V.P. has its solution approaching zero as $t \to \infty$:

$$y'' - y' - 2y = 0; y(0) = y_0, y'(0) = 2.$$

4 pts.

b) Check your work, i.e. demonstrate that your expression solves the I.V.P. found, and approaches zero as $t \to \infty$. If it doesn't, discover your error and correct.

4 pts.

Solution:

The characteristic equation is

$$0 = r^2 - r - 2 = (r-2)(r+1) \Leftrightarrow r = 2, -1.$$

Only a solution associated with $r = -1$ can approach zero for large times. Thus the desired solution is of the form

$$y = Ce^{-t} \Rightarrow$$
$$y' = -Ce^{-t}.$$

Using the initial data we get

$$y(0) = y_0 = C,$$

and then

$$y'(0) = 2 = -C = -y_0 \Leftrightarrow y_0 = -2.$$

4.

i. Let $f = f(t)$ and $g = g(t)$ be two continuously differentiable functions on an interval I, and let $W = W(t)$ be their Wronskian. Which of the following is always correct? Choose as many as are correct.

a) If f and g are linearly dependent on I, then $W(t) \neq 0$ for all $t \in I$.
b) If f and g are linearly dependent on I, then $W(t) = 0$ for all $t \in I$.
c) If $W(t) = 0$ for all $t \in I$, then f and g are linearly dependent on I.
d) If f and g are linearly independent on I, then $W(t) \neq 0$ for all $t \in I$.
e) If f and g are linearly independent on I, then $W(t) = 0$ for all $t \in I$.
f) f and g are linearly dependent on I if and only if $W(t) = 0$ for all $t \in I$.

5 points

Solution:

The correct answer is b). A proof of b) that generalizes to many functions in a natural way is as follows: if f and g are linearly dependent on I, then, by definition, there exists $(k_1, k_2) \neq (0,0)$ such that $k_1 f + k_2 g$ is the zero function on I. Since f and g are differentiable there, $k_1 f' + k_2 g'$ is also the zero function on I. Thus we have that, for each $t \in I$, the system

$$M(t)\mathbf{k} := \begin{bmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (0.26)$$

has nontrivial solution $\mathbf{k} = (k_1, k_2)^T$ (independent of t). But then it must be then that $M(t)$ is not invertible for each such t, i.e. $W(t) := \det M(t) = 0$ for each such t.

ii. Fill in the blanks in the following theorem: If y_1 and y_2 are two solutions of the differential equation $L[y] = y'' + p(t)y' + q(t)y = 0$, then _______ is also a solution for any value of the constants C_1 and C_2.

5 points

Answer:

$C_1 y_1 + C_2 y_2$.

iii. Fill in the blanks in the following theorem: If y_1 and y_2 are two solutions of the differential equation $L[y] = y'' + p(t)y' + q(t)y = 0$, and if there is a point t_0 where the _______of y_1 and y_2 is _______, then the family of solutions $y = C_1 y_1 + C_2 y_2$ with arbitrary coefficients C_1 and C_2 includes _______solution of the differential equation.

9 points

Answers:

Wronskian, non-zero, every.

iv. Fill in the blanks in the following theorem: Consider the differential equation $L[y] = y'' + p(t)y' + q(t)y = 0$, where the coefficient functions $p(t)$ and $q(t)$ are _______. Choose a point t_0 in the interval I . Let y_1 be the solution of the equation that also satisfies the initial conditions

$$y_1(t_0) = 1, \text{ and } y_1{}'(t_0) = 0$$

and let y_2 be a solution that satisfies the initial conditions

$$y_2(t_0) = 0, \text{ and } y_2'(t_0) = 1.$$

Then y_1 and y_2 form a _______ set of solutions of the differential equation on I .

6 points

Answers:

Continuous (in some interval I), fundamental.

5. Solve the I.V.P.

$$y'' - 2y' + 37y = 0, \quad y(0) = 0, \quad y'(0) = 1. \qquad (0.27)$$

10 points

Solution:

The characteristic equation and its roots are as follows:

$$\begin{aligned} 0 &= r^2 - 2r + 37 = (r-1)^2 + 6^2 \\ &= (r-1-6i)(r-1+6i) \\ &\Leftrightarrow r = 1 \pm 6i. \end{aligned}$$

Thus, as per the usual arguments, a real-valued representation of the general solution of the equation is

$$y = (C_1 \cos 6t + C_2 \sin 6t)e^t .$$

The initial data dictates that

$$\begin{aligned} y(0) &= (C_1 \cos 6\cdot 0 + C_2 \sin 6\cdot 0)e^0 = C_1 = 0, \text{ and} \\ y'(0) &= \left[(C_1 + 6C_2)\cos 6\cdot 0 + (C_2 - 6C_1)\sin 6\cdot 0\right]e^0 \\ &= C_1 + 6C_2 = 1 \Leftrightarrow C_2 = 1/6 - 1/6C_1 = 1/6. \end{aligned}$$

Thus the solution to the I.V.P. is

$$y = (0\cdot \cos 6t + 1/6 \sin 6t)e^t = \frac{e^t \sin 6t}{6}.$$

6.

a) Find the value of y_0 so that the following I.V.P. 's solution $y(t)$ has the property that $\lim_{t\to+\infty} \left|e^{2t} y(t)\right| < \infty$:

$$y'' + 4y' + 4y = 0; y(0) = y_0, y'(0) = 2.$$

4 points

b) Check your work, i.e. demonstrate that your expression solves the I.V.P. found, and satisfies $\lim_{t\to+\infty} \left|e^{2t} y(t)\right| < \infty$. If it doesn't, discover your error and correct.

4 points

Solution:

The characteristic equation is

$$0 = r^2 + 4r + 4 = (r+2)(r+2) \Leftrightarrow r = -2, -2.$$

Only a solution of the form

$$y(t) = Ae^{-2t} + Bte^{-2t} = Ae^{-2t} + 0 \cdot te^{-2t} = Ae^{-2t} \qquad (0.28)$$

can give the desired finite limit. Thus

$$y = Ae^{-2t} \Rightarrow y' = -2Ae^{-2t}.$$

Using the initial data we get

$$y(0) = y_0 = A,\ y'(0) = 2 = -2A = -2y_0,$$

whence

$$y_0 = -1.$$

7. Given that $y_1(t) = t^{-1}$ is a solution of

$$2t^2 y'' + 3ty' - y = 0,\ t > 0,$$

find a second, linearly independent solution $y_2(t)$.

10 Points

Solution

The variation of parameter-like ansatz $y = y_2 = y_1 u$ (which leads to a reduction of order), gives

$$\begin{aligned}
0 &= 2t^2 y'' + 3ty' - y = 2t^2 \left(y_1 u\right)'' + 3t\left(y_1 u\right)' - y_1 u \\
&= 2t^2 \left(t^{-1}u\right)'' + 3t\left(t^{-1}u\right)' - t^{-1}u = 2t^2\left(t^{-1}u' - t^{-2}u\right)' + 3t\left(t^{-1}u' - t^{-2}u\right) - t^{-1}u \\
&= 2t^2\left(t^{-1}u'' - 2t^{-2}u' + 2t^{-3}u\right) + 3u' - 3t^{-1}u - t^{-1}u \\
&= 2tu'' - 4u' + 4t^{-1}u + 3u' - 3t^{-1}u - t^{-1}u \\
&= 2tu'' - u' \\
&\Leftrightarrow
\end{aligned}$$

$$\frac{u''}{u'} = \frac{1}{2t} \Leftarrow \ln|u'| = \frac{1}{2}\ln\left(\frac{3}{2}\right)^2 t \Leftarrow u' = \frac{3}{2}t^{1/2} \Leftarrow u = t^{3/2}.$$

Thus $y_2 = y_1 u = t^{-1} t^{3/2} = t^{1/2}$ is a second, linearly independent solution.

8. Find ***a*** solution of the following linear, constant coefficient but non-homogeneous differential equation *by the method of undetermined coefficients*.

$$y'' - 3y' + 2y = te^{2t} \tag{0.29}$$

15 Points

Solution

The function space spanned by the r.h.s. of (0.29) and all its derivatives has basis $\left\{te^{2t}, e^{2t}\right\}$, so that, barring "resonance" of either of these terms, a solution of (0.29) is in the two parameter family

$$y = \left(At + B\right)e^{2t} \tag{0.30}$$

But since the characteristic polynomial of the homogeneous equation indicates $y = Be^{2t}$ is in the solution space of that equation, by the usual "boosting then maintaining linear independence arguments" we must exchange (0.30) for

$$\begin{aligned} y &= \left(At^2 + Bt + 0\right)e^{2t} \Rightarrow \\ y' &= \left(2At^2 + \left(2A + 2B\right)t + B\right)e^{2t} \Rightarrow \\ y'' &= \left(4At^2 + \left(4A + 4A + 4B\right)t + 2A + 2B + 2B\right)e^{2t} \\ &= \left(4At^2 + \left(8A + 4B\right)t + 2A + 4B\right)e^{2t}, \end{aligned} \tag{0.31}$$

i.e.

$$\begin{aligned} 2y &= \left(2At^2 + 2Bt + 0\right)e^{2t}, \\ -3y' &= \left(-6At^2 + \left(-6A - 6B\right)t - 3B\right)e^{2t}, \\ y'' &= \left(4At^2 + \left(8A + 4B\right)t + 2A + 4B\right)e^{2t}, \Rightarrow \\ y'' - 3y' + 2y &= \left(2At + 2A + B\right)e^{2t} = te^{2t} \end{aligned} \tag{0.32}$$

the last from demanding (0.29). So then by linear independence over intervals of time t we must have

$$\begin{aligned} 2A = 1, 2A + B = 0 \Leftrightarrow \\ A = 1/2, B = -1. \end{aligned} \tag{0.33}$$

Thus a solution of (0.29) is (by (0.31))

$$y = \left(At^2 + Bt + 0\right)e^{2t} = \left(\frac{1}{2}t^2 - t\right)e^{2t}\,, \tag{0.34}$$

as can be checked:

$$\begin{aligned}
&y = \left(\frac{1}{2}t^2 - t\right)e^{2t} \Rightarrow \\
y'' - 3y' + 2y &= \left(\left(\frac{1}{2}t^2 - t\right)e^{2t}\right)'' - 3\left(\left(\frac{1}{2}t^2 - t\right)e^{2t}\right)' + 2\left(\frac{1}{2}t^2 - t\right)e^{2t} \\
&= \left(\left(2\frac{1}{2}t^2 - 2t + t - 1\right)e^{2t}\right)' - 3\left(2\frac{1}{2}t^2 - 2t + t - 1\right)e^{2t} + \left(t^2 - 2t\right)e^{2t} \qquad (0.35) \\
&= \left(\left(t^2 - t - 1\right)e^{2t}\right)' - 3\left(t^2 - t - 1\right)e^{2t} + \left(t^2 - 2t\right)e^{2t} \\
&= \left(2t^2 - 2t - 2 + 2t - 1\right)e^{2t} + \left(-3t^2 + 3t + 3\right)e^{2t} + \left(t^2 - 2t\right)e^{2t} \\
&= \left(2t^2 - 3\right)e^{2t} + \left(-2t^2 + t + 3\right)e^{2t} = te^{2t}.
\end{aligned}$$

9. Find the *general solution* of the following linear but non-homogeneous differential equation *by the method of variation of parameters*. Do not use the (memorized) formula/theorem (involving a Wronskian), rather generate the relevant version of the formula afresh by using the “D’Alembert-like” ansatz that leads to that formula. (Also, do not use the method of undetermined coefficients.)

$$y'' - 3y' + 2y = e^{2t} \tag{0.36}$$

15 points

<u>Solution</u>

The characteristic equation of the homogeneous version of the constant coefficient differential equation (0.36) is

$$0 = r^2 - 3r + 2 = (r-1)(r-2) \tag{0.37}$$

so that the general solution of the corresponding homogeneous equation is

$$y = Ae^{t} + Be^{2t}\,, \tag{0.38}$$

where A and B are independent of t. But allowing the parameters A and B to vary with t in (0.38), we have also an ansatz there for the solution of the non-homogeneous equation(0.36): with such an ansatz one immediately has

$$y' = Ae^t + 2Be^{2t} + \left(e^t A' + e^{2t} B'\right). \tag{0.39}$$

But this ansatz is "initially consistent with A and B independent of t" if we choose here that

$$e^t A' + e^{2t} B' = 0, \tag{0.40}$$

so that then (0.39) becomes simply

$$y' = Ae^t + 2Be^{2t}. \tag{0.41}$$

Differentiating (0.41) gives

$$y'' = Ae^t + 4Be^{2t} + \left(e^t A' + 2e^{2t} B'\right). \tag{0.42}$$

Combining these derivatives with the appropriate weights (dictated by the differential equation) we get the ledger

$$\begin{aligned} 2y &= 2Ae^t + 2Be^{2t} \\ -3y' &= -3Ae^t - 6Be^{2t} \\ +1y'' &= Ae^t + 4Be^{2t} + \left(e^t A' + 2e^{2t} B'\right). \end{aligned} \tag{0.43}$$

and from which it is clear that the differential equation demands that

$$e^t A' + 2e^{2t} B' = e^{2t}. \tag{0.44}$$

Combining this with the "consistency ansatz" (0.40) we get

$$\begin{bmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{bmatrix} \begin{bmatrix} A' \\ B' \end{bmatrix} = \begin{bmatrix} 0 \\ e^{2t} \end{bmatrix}, \tag{0.45}$$

which implies that

$$A' = \frac{\begin{vmatrix} 0 & e^{2t} \\ e^{2t} & 2e^{2t} \end{vmatrix}}{\begin{vmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{vmatrix}} = \frac{-e^{4t}}{e^{3t}} = -e^t, \quad B' = \frac{\begin{vmatrix} e^t & 0 \\ e^t & e^{2t} \end{vmatrix}}{\begin{vmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{vmatrix}} = \frac{e^{3t}}{e^{3t}} = 1. \tag{0.46}$$

Solutions to (0.46) include the pair $A = -e^t$, $B = t$, so that a solution to (0.36) is, according to (0.38),

$$y = Ae^t + Be^{2t} = -e^t e^t + te^{2t} = te^{2t} - e^{2t} = (t-1)e^{2t}, \tag{0.47}$$

and the general solution to (0.36) is

$$
\begin{aligned}
y &= c_1 e^t + c_2 e^{2t} + (t-1)e^{2t} \\
&= d_1 e^t + d_2 e^{2t} + te^{2t},
\end{aligned}
\tag{0.48}
$$

where $d_1 = c_1$ and $d_2 = c_2 - 1$ are (truly) constants now.

10. A 5 kilogram mass stretches a (linear, Hooke's law) spring 1/5 meter. If the mass is set in motion from the equilibrium position at 3 meters per second *upward* at time $t=0$, and there is no damping, determine the displacement $u(t)$ of the mass *above* the equilibrium position at any subsequent time t. Use that the acceleration of gravity is 49/5 meters per second per second.

10 points

<u>Solution</u>

The relevant version of Newton's second law is

$$
0 = mu'' + ku = 5\text{kg}u'' + ku \,. \tag{0.49}
$$

Here we may determine the spring constant k from

$$
k = F/\Delta x = mg/\Delta x = \frac{5\text{kg}\left(49/5\right)\dfrac{\text{m}}{s^2}}{1/5\text{m}} = 5\cdot 7^2\,\text{kg}/s^2, \tag{0.50}
$$

so that (0.49) is

$$
0 = 5\text{kg}u'' + 5\cdot 7^2\,\text{kg}/s^2 u \Leftrightarrow 0 = u'' + 7^2/s^2 u \,. \tag{0.51}
$$

Rendering (0.51) unit-less, by measuring time in seconds, this is

$$
0 = u'' + 7^2 u \,, \tag{0.52}
$$

the general solution to which being

$$
u = A\cos(7t) + B\sin(7t) \,. \tag{0.53}
$$

The initial data specifies that

$$
\begin{gathered}
u(0) = 0 = A, u'(0) = 3 = 7B \\
\Leftrightarrow \\
A = 0, B = 3/7,
\end{gathered}
\tag{0.54}
$$

so that the required solution to the initial value problem is

$$
u = A\cos(7t) + B\sin(7t) = \left(3/7\right)\sin(7t). \tag{0.55}
$$

In (0.54) we used that $u'(0) = +3$ rather than $u'(0) = -3$ so that the solution $u = u(t)$ in (0.55) gives, as required by the question, the displacement of the mass *above* the equilibrium position at any time t subsequent to the initial excitation in which it was imparted a velocity of 3 meters per second *upwards*.

11. Determine the steady state response of the damped and driven system

$$y'' + 2y' + 5y = \cos(t). \tag{0.56}$$

Make sure the solution ***is*** the steady state response.

8 points

<u>Solution</u>

Just like in the method of undetermined coefficients we look for solutions of the form

$$y = A\cos(t) + B\sin(t) \tag{0.57}$$

realizing other solutions are in fact damped to zero in the long run: the characteristic polynomial is

$$r^2 + 2r + 5 = (r+1)^2 + 2^2 = (r+1+2i)(r+1-2i) \tag{0.58}$$

whose roots have negative real part. For solutions of the form (0.57) we have

$$\begin{aligned} y &= A\cos(t) + B\sin(t) \Rightarrow \\ y' &= B\cos(t) - A\sin(t) \Rightarrow \\ y'' &= -A\cos(t) - B\sin(t), \end{aligned} \tag{0.59}$$

i.e.

$$\begin{gathered} 5y = 5A\cos(t) + 5B\sin(t), \\ 2y' = 2B\cos(t) - 2A\sin(t), \\ y'' = -A\cos(t) - B\sin(t), \\ \Rightarrow \\ y'' + 2y' + 5y = (4A + 2B)\cos(t) + (-2A + 4B)\cos(t) = \cos(t) \\ \Leftrightarrow \\ 4A + 2B = 1, \ \ -2A + 4B = 0 \Leftrightarrow A = 2B, 10B = 1 \Leftrightarrow \\ A = 1/5, B = 1/10 \Rightarrow \\ y = \frac{1}{5}\cos(t) + \frac{1}{10}\sin(t). \end{gathered} \tag{0.60}$$